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Comparison of Local and Global Approximations for Reliability Estimation

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An approach based on approximation concepts to estimate system reliability is presented. Limit-state functions are approximated explicitly in terms of the random variables of the system. These approximations are then used in conjunction with Monte Carlo simulation to estimate system reliability. Local and global approximations are studied and compared in terms of efficiency and accuracy for their potential use in reliability-based optimization. Example problems are presented to illustrate the ideas set forth.

Introduction

DESIGN optimization via general nonlinear mathematical programming techniques has been widely used for engineering design during the last decade. Traditionally, the objective and constraint functions, load conditions, failure modes, structural parameters, and design variables are treated in a deterministic manner. In general, for a structural design problem involving uncertainties, optimized structures using a deterministic approach may have a higher failure probability than unoptimized structures since they are usually more sensitive to off-design conditions. Therefore, to ensure a certain margin of safety for the design, it is necessary to introduce reliability constraints to achieve a balance between cost and safety for the optimal design. In deterministic optimization this is done by introducing safety factors, but by doing this, the reliability of the final design is never known. Design optimization of structures subject to reliability requirements is generally regarded as the ultimate goal of any design procedure. The basic approach in reliability-based structural optimization is to minimize the total cost or weight of the structure for a specified set of constraints on the overall system reliability or probability of failure. Uncertainties are introduced by using probabilistic models in which structural parameters and/or design variables are considered random or stochastic.

The main problem encountered, when reliability constraints are introduced, is to find an efficient way, from a numerical point of view, to evaluate the system reliability. Moreover, since in an optimization environment these constraints and their sensitivities are evaluated several times, reliability estimation must be computationally inexpensive while preserving a degree of accuracy that makes the optimization results meaningful. The two basic approaches to estimate system reliability are 1) the safety index approach[1] and 2) Monte Carlo simulation.[2] The safety index approach is based on linearized response surfaces and provides accurate approximations for system reliability depending on the curvature of the failure surface about the linearization point. Monte Carlo simulation, on the other hand, gives in general accurate approximations but is computationally very expensive.

Approximations have been used for reliability estimation to alleviate the heavy computational burden associated with numerical simulation. The safety index approach is, in fact, a simple approximation technique in which the failure surface is approximated linearly. In this context, research efforts have been made to minimize the error induced in the approximation by finding the best linearization point, like, for example, the most probable point.[3–5] Approximations for the response surface using higher-order polynomials or other nonlinear functions has received much attention.[6–8] Most of these methods approximate the failure surface in the presence of multiple failure modes using least-squares fit and then use simulation or safety index to estimate reliability or failure probability. Least-squares fit techniques require multiple analyses to construct the approximation and usually fail to capture the nonsmoothness of the response surface because of transitions from one failure mode to another. It is commonly viewed that approximating each failure mode separately is prohibitively expensive since the computational effort to solve for the coefficients in the approximations becomes extremely high with the number of variables.

In this paper, the idea of approximation concepts,[9] commonly used in deterministic structural optimization, is extended to the nondeterministic case. Such an attempt is presented in Ref. 10. In the classical approximation concepts approach to structural synthesis, the optimization is carried out by generating and solving a sequence of explicit approximate optimization problems. In this context, accurate approximations for the objective and constraint response functions are constructed by approximating intermediate response quantities in terms of selected intermediate variables. These approximations are based on first-order information and have, in general, a high degree of accuracy in the neighborhood of their base design expansions. The basic idea in Ref. 10 was to approximate each failure surfaces locally and then use Monte Carlo simulation with the approximate model to estimate system reliability.

The accuracy in the system reliability estimation using this idea is strongly dependent on the degree of nonlinearity in the relation between the response functions and system parameters, and the variance of the random variables, since, for larger deviations, the approximations have to be accurate over a wider range of the design space. The main advantage of the method is that since individual failure modes are approximated separately, the nonsmoothness associated to the response surface is not encountered and only one analysis of the system is required to construct the approximate state-limit functions. In this respect, introducing approximations that are devised to approximate the failure surfaces in a global sense, called global approximations in this work, can, in general, improve the accuracy of the system reliability estimation since they reflect the average behavior of the response functions over a wide range.

In what follows, a comparison of local and global approximations for system reliability estimation is presented. First, the general structural synthesis problem is defined and approximation concepts are introduced in the context of reliability-based optimal design. Finally,

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some example problems are presented to numerically compare both types of approximations for reliability evaluation.

Problem Statement

For a general structural synthesis problem, the constraints are expressed in terms of bounds for displacements, stresses, frequencies, etc. All of these constraints depend on the design variables and fixed structural parameters that, for reliability constraints, are considered random variables. The set of safe or feasible designs can then be expressed as

$$\Omega = \{(y, p) \mid g_j(y, p) \leq 0, \quad j = 1, \ldots, m\} \tag{1}$$

where $y(y_i, i = 1, \ldots, n_y)$ represents the vector of design variables and $p(p_j, j = 1, \ldots, n_p)$ is the vector of structural parameters, both of them modeled as random variables, and the functions $g_j(y, p)$ are deterministic structural responses.

With the previous notation, a reliability-based structural synthesis problem can be written as a nonlinear mathematical optimization problem of the form

$$\text{Min } f(y, p) \qquad \text{s.t.} \qquad \text{Prob}[(y, p) \in \Omega] \geq \bar{P} \tag{2}$$

where $\bar{P}$ is a user-specified level of reliability for the system.

The main computational effort in the solution of this problem is associated with the evaluation of the functions $g_j(y, p)$ and the calculation of the probability. The evaluation of $g_j(y, p)$ requires a structural analysis of the structure at each instance (y, p) and the exact evaluation of the corresponding probability requires the numerical evaluation of a multiple integral.

In an optimization environment, the system reliability function has to be evaluated several times before a near-optimal design can be obtained. Thus, direct application of Monte Carlo simulation is impractical for moderate sized problems even though it gives the best estimate. The safety index approach is less intensive computationally, since the evaluation of the reliability index β requires only one analysis at the best linearization point, but the problems of approximating the response surface and obtaining the linearization point still remain. At the same time, this approach usually suffers from questions of accuracy since the quality of the approximations is strictly dependent on the degree of variation of the random variables and the correlation between failure modes.

Approximation concepts provide a powerful tool to overcome the burden of the evaluation of $g_j(y, p)$. The functions $g_j(y, p)$ are approximated locally or globally for a certain region in the (y, p) space [say $\tilde{g}_j(y, p)$]. Using these approximate functions, an approximation for Ω is constructed ($\tilde{\Omega}$) that is now explicit in both design variables and parameters. The approximate reliability constraint is then given by

$$\text{Prob}[(y, p) \in \tilde{\Omega}] \geq \bar{P} \tag{3}$$

The evaluation of Eq. (3) still requires the evaluation of a multiple integral, which is very costly depending on the number of independent random variables. But, since all approximate surfaces are explicit, using a Monte Carlo simulation technique with $\tilde{\Omega}$ is feasible from a computational point of view. In this work, global approximations for the response functions are developed and their performance is compared with reliability estimation based on local approximations.

Approximation Concepts

The response functions involved in the definition of the limit-state surfaces, that is, the functions $g_j(y, p)$, $j = 1, \ldots, m$, are, in general, given explicitly in terms of intermediate response quantities, which on the other hand are implicit nonlinear functions of the system parameters. For example, consider the stress in a symmetric beam subject to axial loads and bending moment,

$$\sigma = P/A + Mc/I \tag{4}$$

The axial load P and bending moment M are implicit functions of the structural stiffness, which in this case, for simplicity, are represented by A and I. Defining P and M as intermediate response quantities, it is seen from Eq. (4) that σ is explicit in these quantities. The cross-sectional properties A, I, and c are explicit functions of the cross-sectional dimensions (e.g., height, width), which are the actual design variables. Generalizing the concept, the jth limit-state surface can be written as

$$g_j(y, p) = h_j(R, x, y, p) \leq 0 \tag{5}$$

where $R_i, i \in I$ denote the intermediate response quantities and $x_j, j \in J$ denote the intermediate system parameters. It is assumed that 1) h_j is explicit in R, x, y, p; 2) $R_i(x), i \in I$ are implicit functions of x; and 3) $x_j(y, p), j \in J$ are explicit functions of y and p.

Approximations are constructed by approximating the intermediate response quantities R_i explicitly in terms of the intermediate variables x, to give $\tilde{R}_i(x), i \in I$. With these approximate quantities, the approximate limit-state surface is given by

$$\tilde{g}_j(y, p) = h_j\{\tilde{R}[x(y, p)], x(y, p), y, p\} \leq 0 \tag{6}$$

Applying these concepts for the beam stress given in Eq. (4), the force P and moment M are approximated in terms of A and I giving $\tilde{P}(A, I)$ and $\tilde{M}(A, I)$. To complete the approximation A and I are expressed explicitly in terms of the actual variables. To illustrate the idea, assume that the cross section is rectangular with height h and width b, then

$$A(b, h) = bh \tag{7a}$$

$$I(b, h) = \tfrac{1}{12}bh^3 \tag{7b}$$

Then, the approximate stress is given by

$$\tilde{\sigma}(b, h) = \frac{\tilde{P}[A(b, h), I(b, h)]}{A(b, h)} \pm \frac{\tilde{M}[A(b, h), I(b, h)]}{I(b, h)} \frac{h}{2} \tag{8}$$

This approximation captures most of the inherent nonlinearities of the actual function and provides, in general, very accurate approximate stresses.[11] Note that, if a first-order approximation is chosen for P and M, the approximate stress is exact for statically determinate structures.

For the general case, the approximations for $R_i, i \in I$ in Eq. (6) (denoted by $\tilde{R}_i$) can either be local, using information at a given base design, or global to reflect the functional behavior over a wider range of the design space. Local approximations are exact at the base design, and the error increases with the distance to it. Global approximations give, on the other hand, a more accurate description of the function over a wide range but are not necessarily exact at any point, and usually at a higher computational cost. In what follows, both approaches will be examined from the point of view of accuracy for reliability estimation and computational effort associated to the construction of the approximations.

Local Approximations

The approximations for the intermediate response quantities are constructed using local information about a base line design. For example, $\tilde{R}_i(x)$ can be written in general form as

$$\tilde{R}_i(x) = \sum_{j=1}^{N} R_{ij}(x_0, x) \tag{9}$$

where $x_0 = x(y_0, p_0)$, (y_0, p_0) is the base line design for the system parameters and $R_{ij}(\cdot)$ is a multivariate polynomial expansion of jth order about x_0. In this paper, the base line design is taken as the mean value for random variables.

For the case of static problems, one of such intermediate response quantities is the solution of the static equilibrium equation of the structural system, that is, the solution of the equation

$$Ku = f \tag{10}$$

where u represents the dependent variable, K is the stiffness matrix, and f represents the external forces. As previously mentioned, for all but trivial cases, this function is available only in an algorithm sense, e.g., a finite element code.

The dependent variable u can now be approximated locally about a base design, using first- or higher-order information about such point. The most simple first-order approximation corresponds to a Taylor expansion in terms of the set of system parameters. In that case, approximation for the dependent variable u takes the form

$$\tilde{u}(x) = u(x_0) + \sum_{i=1}^{N}(x_i - x_{i0})\frac{\partial u}{\partial x_i}(x_0) \tag{11}$$

where x_0 in the nominal or baseline value for the set of intermediate parameters. As mentioned earlier, this approximation has a high degree of accuracy only in the neighborhood of its base design. To enhance the global quality of the approximation, intermediate variables as well as reciprocal or hybrid variables have been used extensively in the context of structural synthesis.[11,12]

For the case of reciprocal variables, the approximation is

$$\tilde{u}(x) = u(x_0) - \sum_{i=1}^{N} x_{i0}^2\left(\frac{1}{x_i} - \frac{1}{x_{i0}}\right)\frac{\partial u}{\partial x_i}(x_0) \tag{12}$$

When the response function is highly nonlinear, second-order approximations may be required. In the case of direct variables, the approximation for the dependent variable u is given by

$$\tilde{u}(x) = u(x_0) + \sum_{i=1}^{N}(x_i - x_{i0})\frac{\partial u}{\partial x_i}(x_0) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - x_{i0})(x_j - x_{j0})\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \tag{13a}$$

and the second-order approximation in terms of reciprocal variables is given by

$$\tilde{u}(x) = u(x_0) - \sum_{i=1}^{N} x_{i0}^2\left(\frac{1}{x_i} - \frac{1}{x_{i0}}\right)\frac{\partial u}{\partial x_i}(x_0) + \sum_{i=1}^{N} x_{i0}^3\left(\frac{1}{x_i} - \frac{1}{x_{i0}}\right)^2\frac{\partial u}{\partial x_i}(x_0) + \sum_{i=1}^{N}\frac{x_{i0}^4}{2}\left(\frac{1}{x_i} - \frac{1}{x_{i0}}\right)^2\frac{\partial^2 u}{\partial x_i^2}(x_0) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} x_{i0}^2 x_{j0}^2\left(\frac{1}{x_i} - \frac{1}{x_{i0}}\right)\left(\frac{1}{x_j} - \frac{1}{x_{j0}}\right)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \tag{13b}$$

Note that the nominal value of the dependent variable $[u(x_0)]$ corresponds to the solution of a standard equilibrium equation, whereas the derivatives of such a function are the solution of the same equation with different right-hand sides. The solution of these equations can be computed very easily using the finite element method. Once the intermediate response functions have been approximated, the approximate failure surfaces are given explicitly in terms of the set of system parameters. The approximate system reliability is obtained by combining these approximations with simulation techniques, which is now feasible from a computational point of view, because of the explicitness of the functions $\tilde{g}_j(y, p)$, $j = 1, \ldots, m$.

Global Approximations

To introduce the concept of global approximations, consider again the static case. If the selection of the intermediate design variables is such that the stiffness properties of the structural system are multivariate polynomial functions of x (intermediate variables), then the stiffness matrix can be written as

$$K(x) = K_0 + \sum_{i=1}^{N} x_i K_i + \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j K_{ij} + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{N} x_i x_j x_l K_{ijl} + \cdots \tag{14}$$

where the matrices K_0, K_i, K_{ij}, K_{ijl}, ..., are the coefficients of the stiffness matrix expansion. If the stiffness properties are not polynomial functions of the set of intermediate design variables, then an expansion similar to Eq. (14) can be defined by a least-squares fit between the multivariate expansion and the exact dependence of the stiffness matrix on x.

Equation (10) shows the relationships between the system response and the system parameters. To solve the equation of equilibrium, the dependent variable u is expanded in terms of a complete set of analytical functions. In this paper, a set of orthogonal polynomials $\prod_{i=1}^{N} L_{q_i}(x_i)$, $q_i = 0, 1, \ldots, i = 1, \ldots, N$, is used as a basis for the expansion of the dependent variables. The polynomials are orthogonal in the inner product space defined by the mean operation, that is,

$$E[L_q(x_i)L_s(x_i)] = \delta_{qs} \tag{15}$$

where $E(\cdot)$ is the expectation operation and δ_{qs} is the Kronecker delta. Expansions in terms of direct and reciprocal variables are considered here. The selection for the type of expansion depends on the functional dependence of the response function on the set of intermediate design variables. For some responses, expansions in terms of reciprocal variables are more accurate than those that use direct variables and vice versa. Some comparisons are presented in the Example Problems section.

The expansion of the vector u is written as

$$\tilde{u}(x) = \sum_{i\geq 0} u_i(x) \tag{16}$$

where $u_i(\cdot)$, $i \geq 0$ represents a polynomial expansion of ith order given by

$$u_0(x) = u_{0,\ldots,0} \tag{17}$$

and

$$u_i(x) = \sum_{|q|=i} u_{q_1,\ldots,q_N}\prod_{j=1}^{N} L_{q_j}[\kappa(x_j)], \qquad i \geq 1 \tag{18}$$

where q is the vector with components q_i and $|q|$ stands for the norm of q defined by $|q| = \sum_{i=1}^{N} q_i$, $\kappa(\cdot)$ is either the identity function $[\kappa(x_i) = x_i]$ or the inverse function $[\kappa(x_i) = 1/x_i]$, and $u_{q_1,\ldots,q_N}$ are the unknown coefficients of the expansion. For example, the polynomial expansion of first order $u_1(x)$ is given by

$$u_1(x) = u_{1,0,\ldots,0}L_1[\kappa(x_1)]L_0[\kappa(x_2)]\cdots L_0[\kappa(x_N)] + u_{0,1,\ldots,0}L_0[\kappa(x_1)]L_1[\kappa(x_2)]\cdots L_0[\kappa(x_N)] + \cdots + u_{0,0,\ldots,1}L_0[\kappa(x_1)]L_0[\kappa(x_2)]\cdots L_1[\kappa(x_N)] \tag{19}$$

The higher-order polynomials can be defined in a similar manner directly from Eq. (18).

The system response is fully characterized by the coefficients of the expansion (16). The weighted residual method is used to derive a set of equations for computing the unknown coefficients of the expansion. For this, a residual function is first defined as

$$R_e(x) = K(x)\tilde{u}(x) - f \tag{20}$$

Then the residual function is minimized by forcing it to be orthogonal, with respect to the inner product defined by the mean operation, to the space of functions used in the expansion of the vector of intermediate response quantities. The equations for the unknown coefficients of the expansion are obtained by substituting Eqs. (14) and (16) into the residual function. Orthogonality conditions along with appropriate recursive formulas for the orthogonal basis are then used to derive the set of equations for the coefficients of the expansion.

It can be shown that the basis of Legendre polynomials is particularly appropriate when the system parameters are modeled with uniform distributions, since its orthogonality condition greatly simplifies the derivation of the equation for the unknown coefficients.[13] Similarly, the Hermite polynomials are appropriate for normal random variables and Laguerre polynomials for exponential random

variables. For other probability distributions, suitable basis can also be defined.[14,15] Rearranging the coefficients $u_{q_1,\ldots,q_N}$ into a vector $\hat{u}$, the global system of equations for the unknowns can be written as

$$\hat{K}\hat{u} = \hat{f} \tag{21}$$

where $\hat{K}$ is the generalized stiffness matrix of the system and $\hat{f}$ is the generalized total effective load vector. The particular structure of the system of equations depends on the order of the multivariate expansion of the stiffness matrix, and on the order of the approximation for the dependent variable u (Refs. 13 and 16).

The global system of equations is now solved to find the unknowns of the expansion. The dimension of the generalized stiffness matrix increases fast with the number of design parameters involved in the problem, which may pose problems with the computer storage required for the numerical implementation. Nevertheless, this system is sparse, which reduces both the computational effort and computer storage requirements. In addition, the global system of equations remains symmetric and positive definite (the same properties of the standard stiffness matrix). The system is solved numerically using iterative methods, since they involve the system matrix only in matrix-vector multiplications, allowing an efficient implementation in a vector processing computer environment. Therefore, methods such as Gauss–Seidel, conjugate gradients, or preconditioned conjugate gradients are appropriate for this formulation. Gauss–Seidel iteration is the algorithm used for the numerical examples presented in this paper. Experience has shown that few iterations are needed to provide reasonable accuracy for the numerical examples considered by the authors. Specific values are given in the Example Problems section.

Finally, once the approximation for the dependent variable has been obtained (i.e., once the coefficients have been solved for), the response functions $g_j(y, p)$, $j = 1, \ldots, m$ can be written explicitly in terms of the set of system parameters. As before, because of the explicitness of the approximations, the reliability can be evaluated in an efficient way using simulation.

Illustrative Example

To illustrate the formulation presented earlier, a structural system defined by truss members is considered. The design variable for member e is selected to be the cross-sectional area A_e that is assumed to be uniformly distributed. Then the elemental stiffness matrix can be written as

$$\bar{K}^e(x_e) = x_e \bar{K}_e \tag{22}$$

where the intermediate design variable x_e is given by $x_e(y_e) = y_e = A_e$ and the matrix $\bar{K}_e$ is defined as

$$\bar{K}_e = E/L_e \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{23}$$

where E is the elastic modulus and L_e is the length of the element.

To assemble the global stiffness matrix of the structural system, the elemental matrices are transformed into a global coordinate system through a transformation matrix. Then, for example, using the standard finite element methodology, the global stiffness matrix $K(x)$ is assembled and written as in Eq. (14). In this particular case, the expansion of the stiffness matrix in terms of the intermediate design parameters takes the form

$$K(x) = \sum_{e=1}^{N_e} x_e K_e \tag{24}$$

where N_e is the number of elements in the structure, and K_e, $e = 1, \ldots, N_e$ are the extended version of the elemental matrices $\bar{K}_e$, $e = 1, \ldots, N_e$, in global coordinates. It is noted that Eq. (24) assumes that the number of design variables is equal to the number of structural elements, that is, $N = N_e$. Similar expressions for the expansion of the stiffness matrix can be obtained for other system parameters and for structural systems modeled by elements other than truss members.

The equations for the coefficients $u_{q_1,\ldots,q_N}$ are obtained as previously described. If the expansion for the dependent variable u is defined in terms of direct variables, then the coefficients are governed by the set of equations

$$\sum_{i=1}^{N} K_i \left[\alpha_{p_i} \Delta_i u_{p_1,\ldots,p_i-1,\ldots,p_N} + \Delta_i^* u_{p_1,\ldots,p_i,\ldots,p_N} + \alpha_{p_i+1}\Delta_i \times u_{p_1,\ldots,p_i+1,\ldots,p_N}\right] = 2^{N/2} \prod_{i=1}^{N} \left[\Delta_i^{\frac{1}{2}} \delta_{p_i 0}\right] f \qquad p_1, \ldots, p_N \geq 0 \tag{25}$$

where $\alpha_{p_i} = p_i/\sqrt{[(2p_i - 1)(2p_i + 1)]}$, δ_{ij} is the Kronecker delta, $\Delta_i = [x_{iu} - x_{il}]/2$, and $\Delta_i^* = [x_{iu} + x_{il}]/2$, with x_{il} being the lower limit of the range of variation of the intermediate design variable x_i and x_{iu} the corresponding upper limit. For example, the equation corresponding to the indices $p_i = 0$, $i = 1, \ldots, N$ takes the form

$$\left(\sum_{i=1}^{N} K_i \Delta_i^*\right) u_{0,0,\ldots,0} + \alpha_1 \{K_1 \Delta_1 u_{1,0,\ldots,0} + K_2 \Delta_2 u_{0,1,\ldots,0} + \cdots + K_N \Delta_N u_{0,0,\ldots,1}\} = \left(2^{N/2} \prod_{i=1}^{N} \Delta_i^{\frac{1}{2}}\right) f \tag{26}$$

On the other hand, if the expansion is given in terms of reciprocal variables, that is,

$$\tilde{u}(x) = \sum_{|q| \geq 0} u_{q_1,\ldots,q_N} \prod_{i=1}^{N} L_{q_i}(1/x_i) \tag{27}$$

then the set of equations for the coefficients becomes (equation corresponding to the set of indices $s_1, \ldots, s_N \geq 0$)

$$\sum_{|q| \geq 0} \sum_{i=1}^{N} K_i u_{q_1,\ldots,q_N} \prod_{j=1, j\neq i}^{N} \left[\alpha_{s_j+1} \Delta_j \delta_{q_j, s_j+1} + \Delta_j^* \delta_{q_j s_j} + \alpha_{s_j} \Delta_j \delta_{q_j, s_j-1}\right] \delta_{q_i s_i} = 2^{N/2} \prod_{i=1}^{N} \Delta_i^{\frac{1}{2}} \left[\alpha_1 \Delta_i \delta_{s_i 1} + \Delta_i^* \delta_{s_i 0}\right] f \qquad s_1, \ldots, s_N \geq 0 \tag{28}$$

where Δ_i and Δ_i^* are now defined as $[1/x_{il} - 1/x_{iu}]/2$ and $[1/x_{il} + 1/x_{iu}]/2$, respectively, and all other terms are as previously defined. The derivation of equations similar to Eqs. (25) and (28) is given in Refs. 16 and 17.

As mentioned before, the structure of the generalized stiffness matrix $\hat{K}$ depends on the order of approximation of u. If a first-order approximation is considered, then the system matrix is given by

$$\hat{K} = \begin{pmatrix} H_0 & H_1 & H_2 & \cdots & H_N \\ H_1 & H_0 & 0 & \cdots & 0 \\ H_2 & 0 & H_0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ H_N & 0 & 0 & \cdots & H_0 \end{pmatrix} \tag{29}$$

where $\hat{u}$ is defined as $\hat{u}^T = \{u_{0,0,\ldots,0}, u_{1,0,\ldots,0}, u_{0,1,\ldots,0}, \ldots, u_{0,0,\ldots,1}\}$ and where

$$H_0 = \sum_{i=1}^{N} \Delta_i^* K_i \tag{30}$$

and

$$H_l = \alpha_1 \Delta_l K_l, \qquad l = 1, \ldots, N \tag{31}$$

for an expansion in terms of direct variables, and

$$H_0 = \sum_{i=1}^{N} K_i \left[\prod_{j=1, j\neq i}^{N} \Delta_j^*\right] \tag{32}$$

Table 1 Portal frame, local approximations, uniform distributions, one failure mode

Coefficient of variation I_1, I_2, %	Probability of failure				
	Monte Carlo, exact	Linear direct variables	Quadratic direct variables	Linear reciprocal variables	Quadratic reciprocal variables
5	7.286×10^{-4}	1.590×10^{-4}	6.631×10^{-4}	7.311×10^{-4}	7.286×10^{-4}
10	4.860×10^{-2}	2.920×10^{-2}	4.544×10^{-2}	5.030×10^{-2}	4.843×10^{-2}
15	1.435×10^{-1}	0.868×10^{-1}	1.314×10^{-1}	1.495×10^{-1}	1.422×10^{-1}

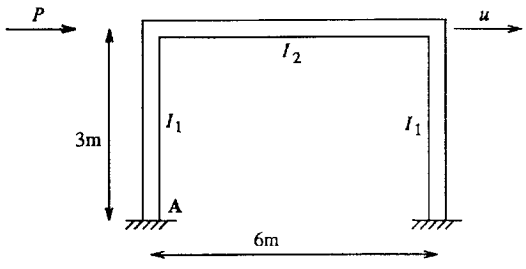


Fig. 1 Portal frame structure.

and

$$H_l = \alpha_1 \Delta_l \sum_{i=1, i\neq l}^{N} K_i \left[\prod_{j=1, j\neq i, j\neq l}^{N} \Delta_j^* \right] \qquad l = 1, \ldots, N \tag{33}$$

for an expansion in terms of reciprocal variables.

The generalized stiffness matrix can be defined in a similar manner if higher-order approximations are considered for the expansion of the dependent variable u. Similar expressions for the set of equations for the coefficients of the expansion $u_{q_1,\ldots,q_N}$ can be obtained if distributions other than uniform are considered.

Example Problems

The following examples have been chosen to compare local and global approximations for reliability evaluation.

Problem 1: Portal Frame

The first example problem involves the portal frame structure shown in Fig. 1. The load P and Young's modulus are considered deterministic with a values of 2000 kgf and 2500 kgf/mm^2, respectively. Two limit states are considered in this example and given by the conditions

$$u \leq U \tag{34a}$$

and

$$\sigma \leq \Sigma \tag{34b}$$

where the degree of freedom u is shown in Fig. 1 and σ is the maximum axial bending stress at point A (see Fig. 1). The terms U and Σ are random variables with coefficient of variation of 5% and with mean values equal to 1.2 times the nominal values of the displacement u and stress σ (solution obtained with mean value for the design parameters), respectively. The moments of inertia I_1 and I_2 are also considered uniform random variables. The mean value for these variables is given by $\bar{I}_1 = 3 \times 10^8$ mm^4 and $\bar{I}_2 = 4 \times 10^8$ mm^4.

Two cases were studied for this problem. In the first case, all random variables are uniform, and in the second case, all random variables are normal. Tables 1–8 show the approximation to the probability of failure using local and global approximations. Linear and quadratic approximations in terms of direct and reciprocal variables are considered.

Monte Carlo simulations were done with a sample size such that the confidence of the estimation is 1%. In all tables, the column labeled Monte Carlo gives the probability of failure using simulation on the exact system, thus these values are considered exact for comparison purposes.

Uniform Distributions

Tables 1 and 2 show the probability of failure of the system considering one failure mode [bound on displacement, see Eq. (34a)]. From the tables it is seen that both local and global approximations have a similar behavior, which is expected for this problem, since the response function is smooth. Comparing the error in the approximation, it is seen that for the same order and choice of variables, the global approximations are more accurate for system reliability estimation. This effect is especially important for larger coefficients of variation of the system parameters. The case of two failure modes [Eqs. (34)] is shown in Tables 3 and 4, which show a similar behavior as in the case of one failure mode. It is important to note that in this case both limit-state surfaces were approximated separately, and thus the correlation between modes is completely captured by the approximation.

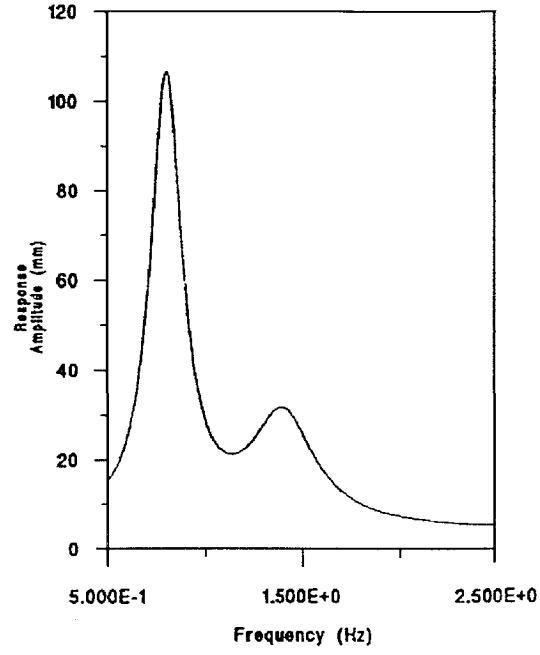


Fig. 2 Steady-state response amplitude for a two-degree-of-freedom system.

Normal Distributions

In this case all random variables were assumed to have a truncated normal distribution. Tables 5 and 6 show the approximation to the probability of failure considering only one failure mode, and Tables 7 and 8 show the case of two failure modes. From these tables, it is seen that in terms of accuracy the situation is similar to the case with uniform distributions. Comparing Tables 1–4 with Tables 5–8, one can observe that the probability of failure for the normal case is smaller than the uniform case, since the distribution is concentrated around the mean value.

Finally, the number of iterations required for the convergence of the Gauss–Seidel method, in the global approximation approach [solution of Eq. (21)], was five for this example problem.

Problem 2: Frequency Response

This second example problem involves the amplitude of the steady-state response $F(\Omega)$ for a two-degree-of-freedom system shown in Fig. 2. The limit state considered in this example is given by the condition

$$F(\Omega) \leq F_0 \tag{35}$$

where F_0 is the threshold level.

Table 2 Portal frame, global approximations, uniform distributions, one failure mode

Coefficient of variation I_1, I_2, %	Probability of failure				
	Monte Carlo, exact	Linear direct variables	Quadratic direct variables	Linear reciprocal variables	Quadratic reciprocal variables
5	7.286×10^{-4}	3.035×10^{-4}	7.127×10^{-4}	7.310×10^{-4}	7.286×10^{-4}
10	4.860×10^{-2}	4.101×10^{-2}	4.845×10^{-2}	4.881×10^{-2}	4.860×10^{-2}
15	1.435×10^{-1}	1.290×10^{-1}	1.443×10^{-1}	1.447×10^{-1}	1.435×10^{-1}

Table 3 Portal frame, local approximations, uniform distributions, two failure modes

Coefficient of variation I_1, I_2, %	Probability of failure				
	Monte Carlo, exact	Linear direct variables	Quadratic direct variables	Linear reciprocal variables	Quadratic reciprocal variables
5	3.115×10^{-3}	0.960×10^{-3}	2.870×10^{-3}	3.145×10^{-3}	3.115×10^{-3}
10	1.069×10^{-1}	0.714×10^{-1}	1.015×10^{-1}	1.089×10^{-1}	1.066×10^{-1}
15	2.306×10^{-1}	1.695×10^{-1}	2.206×10^{-1}	2.346×10^{-1}	2.300×10^{-1}

Table 4 Portal frame, global approximations, uniform distributions, two failure modes

Coefficient of variation I_1, I_2, %	Probability of failure				
	Monte Carlo, exact	Linear direct variables	Quadratic direct variables	Linear reciprocal variables	Quadratic reciprocal variables
5	3.115×10^{-3}	1.625×10^{-3}	3.055×10^{-3}	3.090×10^{-3}	3.115×10^{-3}
10	1.069×10^{-1}	0.951×10^{-1}	1.071×10^{-1}	1.073×10^{-1}	1.069×10^{-1}
15	2.306×10^{-1}	2.292×10^{-1}	2.345×10^{-1}	2.307×10^{-1}	2.306×10^{-1}

Table 5 Portal frame, local approximations, normal distributions, one failure mode

Coefficient of variation I_1, I_2, %	Probability of failure				
	Monte Carlo, exact	Linear direct variables	Quadratic direct variables	Linear reciprocal variables	Quadratic reciprocal variables
5	6.000×10^{-5}	0.000	4.997×10^{-5}	5.996×10^{-5}	5.997×10^{-5}
10	1.765×10^{-2}	0.878×10^{-3}	1.613×10^{-2}	1.835×10^{-2}	1.757×10^{-2}
15	8.260×10^{-2}	4.363×10^{-2}	7.396×10^{-2}	8.608×10^{-2}	8.176×10^{-2}

Table 6 Portal frame, global approximations, normal distributions, one failure mode

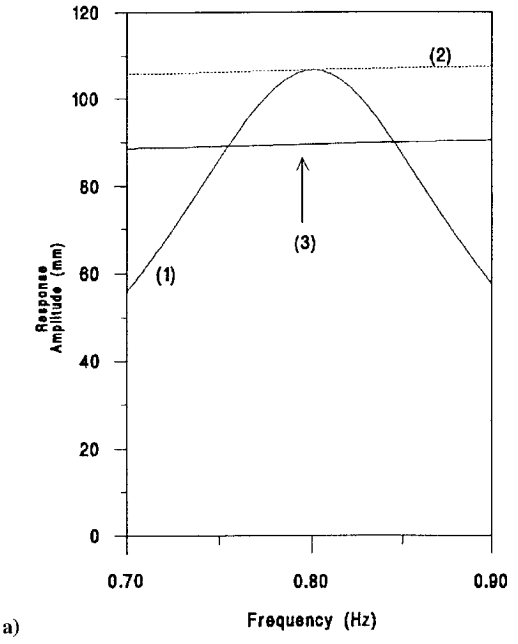
Coefficient of variation I_1, I_2, %	Probability of failure				
	Monte Carlo, exact	Linear direct variables	Quadratic direct variables	Linear reciprocal variables	Quadratic reciprocal variables
5	6.000×10^{-5}	1.000×10^{-5}	5.501×10^{-5}	5.500×10^{-5}	5.997×10^{-5}
10	1.765×10^{-2}	1.425×10^{-2}	1.764×10^{-2}	1.748×10^{-2}	1.766×10^{-2}
15	8.260×10^{-2}	7.672×10^{-2}	8.445×10^{-2}	8.230×10^{-2}	8.261×10^{-2}

Table 7 Portal frame, local approximations, normal distributions, two failure modes

Coefficient of variation I_1, I_2, %	Probability of failure				
	Monte Carlo, exact	Linear direct variables	Quadratic direct variables	Linear reciprocal variables	Quadratic reciprocal variables
5	3.600×10^{-4}	0.900×10^{-4}	3.050×10^{-4}	3.600×10^{-4}	3.600×10^{-4}
10	4.619×10^{-2}	2.574×10^{-2}	4.290×10^{-2}	4.721×10^{-2}	4.609×10^{-2}
15	1.426×10^{-1}	0.947×10^{-1}	1.351×10^{-1}	1.452×10^{-1}	1.421×10^{-1}

Table 8 Portal frame, global approximations, normal distributions, two failure modes

Coefficient of variation I_1, I_2, %	Probability of failure				
	Monte Carlo, exact	Linear direct variables	Quadratic direct variables	Linear reciprocal variables	Quadratic reciprocal variables
5	3.600×10^{-4}	1.500×10^{-4}	3.400×10^{-4}	3.600×10^{-4}	3.600×10^{-4}
10	4.619×10^{-2}	3.921×10^{-2}	4.641×10^{-2}	4.613×10^{-2}	4.621×10^{-2}
15	1.426×10^{-1}	1.485×10^{-1}	1.478×10^{-1}	1.418×10^{-1}	1.427×10^{-1}



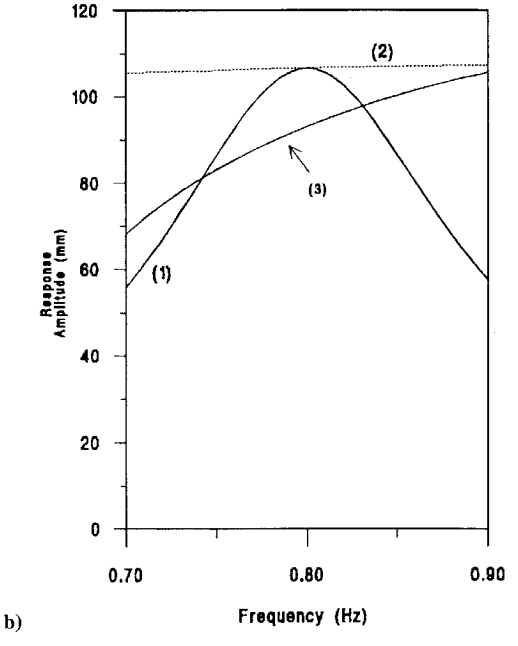
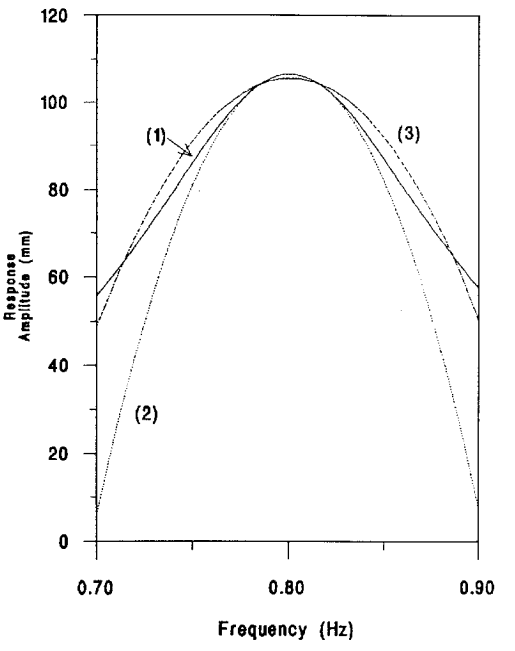


Fig. 3 First-order approximations in the neighborhood of the first resonance peak; (1) exact response, (2) local approximation, and (3) global approximation: a) approximation using direct variables and b) approximation using reciprocal variables.



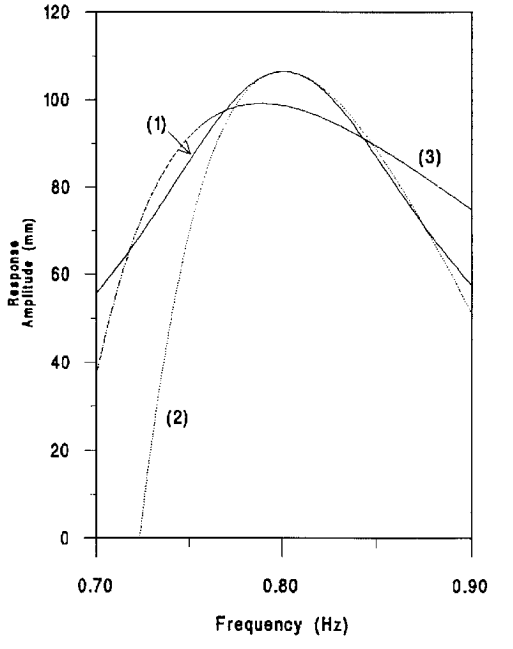


Fig. 4 Second-order approximations in the neighborhood of the first resonance peak; (1) exact response, (2) local approximation, and (3) global approximation: a) approximation using direct variables and b) approximation using reciprocal variables.

The random structural parameter is Ω, the forcing frequency. This variable is considered uniform with mean value 0.8 Hz, which corresponds to the first resonant peak. This case is selected to show that first-order approximations may not be able to capture the behavior of a highly nonlinear response over the desired range and higher-order approximations might be required. The steady-state response corresponds to the solution of a linear system of equations, which is the one used in the global approximation approach. Figure 3 shows both local and global first-order approximations for $F(\Omega)$ in terms of the direct and reciprocal variables. It is clear that these approximations are not able to capture the curvature of the response function, and therefore the system reliability is not well approximated.

Local and global quadratic approximations are shown in Fig. 4. These second-order approximations capture the curvature of the response for both types of expansions (direct and reciprocal variables). It is noted, however, that the approximation obtained by the local expansion is accurate only in the neighborhood of the mean value of the structural parameter. In this case, the overall behavior of the response is better represented by the global expansion in terms of direct variables. Figure 5 shows the exact and approximate system reliability for a coefficient of variation of 10% for Ω. The global approximations give a better estimate even though the approximation for the response itself is less accurate (see Fig. 4). The figure also shows that the system reliability is overestimated when using local approximations. This is interesting, since in a design process overestimating the system reliability can mislead the final design. In this example, the number of iterations required for solving the global system of equations in the global approximation approach was 10. This number of iterations in addition to the one required in the first example indicates that, in general, only a few iterations are needed to get a reasonable accuracy in the results.

Finally, Fig. 6 shows the comparison between the local second-order approximation and a higher-order global approximation, both using direct variables. The global approximation uses the first three

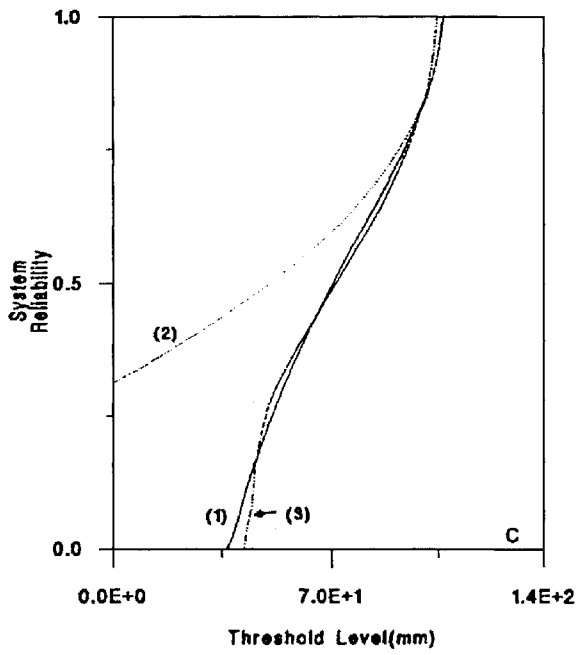


Fig. 5 System reliability using second-order approximation for 10% coefficient for variation for the forcing frequency: (1) exact response, (2) local approximation, and (3) global approximation.

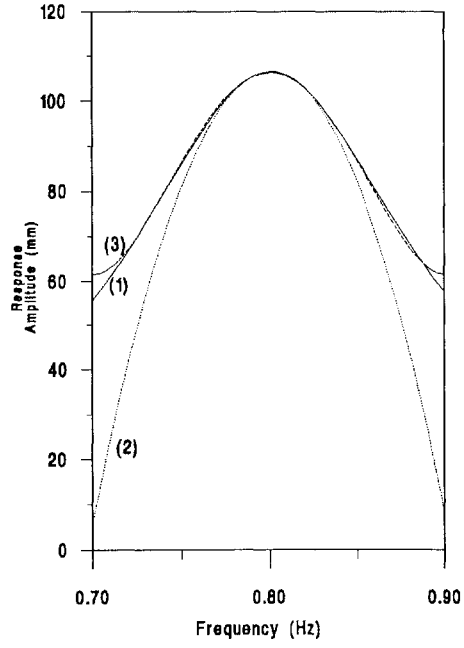


Fig. 6 Response amplitude: (1) exact, (2) second-order local approximation using direct variables, and (3) first three even terms in the global approximation using direct variables.

even terms of the expansion, that is, a constant term, a quadratic term, and a quartic term. Since there are three terms in the expansion, the computational effort is identical to the case of a full quadratic approximation, but, as seen in Fig. 6, accuracy is greatly increased (compare Figs. 4 and 6). On the other hand, if a fourth-degree expansion is to be constructed using a local approximation, all sensitivities up to order three must be computed, independent of which terms are retained for the approximation.

Conclusions

A methodology for estimating system reliability based on approximation concepts has been presented. In this approach, limit-state surfaces are approximated explicitly in terms of the random variables of the system. These approximate surfaces are then used with Monte Carlo simulation to obtain the system reliability. From the numerical point of view, this method is efficient, since the heavy computational effort associated to simulation involves only the evaluation of the explicit approximate limit-state surfaces.

Local and global approximations were introduced and compared for system reliability estimations. Local approximations are very accurate in the neighborhood of the base line design, and thus if the variability of the random variables increases, the quality of the system reliability approximation decreases. On the other hand, global approximations better reflect the average behavior of the limit-state functions over a wider range in the space of system parameters. Therefore, with this type of approximation a more accurate estimate of system reliability is obtained. The numerical problems demonstrate this expected behavior. In particular, the first example problem shows that for the same order and choice of intermediate variables, global approximations give a more accurate system reliability estimation.

The accuracy given by local approximations is strongly dependent on the type of probabilistic distribution and degree of variability of the random variables since points away from the mean value have lower probability. On the other hand, global approximations can be adjusted to different types of distributions by choosing appropriately the polynomials for the expansion and the weighting functions in the weighted residual method. This is an attractive feature since the method is able to capture different conditions of the system and variables.

From a computational point of view, local approximations are less expensive to compute than global approximations. In fact, the computational cost associated to global approximations is proportional to the number of intermediate parameters and the order of the approximation. For example, for a first-order approximation, the system of equations to be solved [see Eq. (29)] has dimension on the order of the number of degrees of freedom times the number of intermediate parameters. But since the system has a sparse structure, it can then be solved very efficiently using iterative methods. For the example problems presented, the number of Gauss–Seidel iterations required for convergence was under 10.

The methodology presented in this paper is able to consider multiple failure modes simultaneously since all limit-state functions are approximated independently, and thus correlation of failure modes is implicitly captured. Moreover, the random variables are not required to be independent for approximation purposes. The correlation of the variables only appears in the simulation stage where all statistical interdependency can be easily incorporated by well-known techniques.

When using these approximations in the context of optimal design, choosing between local or global depends on accuracy and computational effort. In this work it is shown that global approximations improve accuracy but at a higher computational cost. Further numerical research is required to determine when to use one or the other, or even combine local and global approximations depending on the type of random variable, probability distribution, and degree of nonlinearity of the limit-state surface.

This tradeoff must also be evaluated in terms of the number of design cycles required for convergence to the optimal design. It is a known fact that using more accurate approximations reduces the number of design cycles, i.e., number of real structural analyses. Thus, even if the global approximations are more expensive to obtain, the reduction in the number of design cycles can be important enough to justify their use for certain types of problems.

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